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Infinitesimal cubical structure, and higher connections

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Introduction

The purpose of the present note is to experiment with a possible framework for the theory of "higher connections", as have recently become expedient in string theory; however, our work was not so much to find a framework which fits any existing such theory, but rather to find notions which come most naturally of their own accord. The approach is based on the combinatorics of the "first neighbourhood of the diagonal" of a manifold, using the technique and language of Synthetic Differential Geometry (SDG), as in [8], and notably in [11]. The present note may be seen as a sequel to the latter, and also to [14]. The basic viewpoint in [11] is that connections (1-connections) take value in *groupoids* (a viewpoint which goes back to Ehresmann), and that they in effect may be seen as morphisms in the category of reflexive symmetric graphs, noting that any groupoid has an underlying such.

To go beyond this into higher dimensions, one would like to consider some kind of higher groupoids to receive the values of the connections (see e.g. [18]), and we also need a higher dimensional version of the notion of reflexive symmetric graph. This led us to pass into the cubical, rather than into the simplicial world. The passage into this world depends on having a cubical complex associated to any manifold, in analogy with the simplicial complex of "infinitesimal simplices" that one derives out of the first neighbourhood of the diagonal. This latter simplicial complex is known to be the carrier of a theory of "combinatorial differential forms", as in [8], [11], [13], and in [1].

The observation that opens up for a similar cubical complex is that infinitesimal simplices in a manifold canonically give rise to infinitesimal parallelepipeda. This hinges on the possibility of forming affine combinations of the points in an infinitesimal simplex, a possibility first noted in [12], see also [13].

On the algebraic side, the kind of "higher" groupoid which fits the bill are essentially the "ω-groupoids" of Brown and Higgins [2], or their truncation

to some finite dimension, in which case we call them n-cubical groupoids. For $n = 2$ they are the edge-symmetric double groupoids of Brown and Spencer, [3]. In any case, all structure involved (at present) is strict, and no coherence issues are involved.

Besides connection as an infinitesimal notion, we study a corresponding global notion, which is that of holonomy. We relate holonomy of n-connections with integrals of differential n-forms.

This research was partly triggered by some questions which Urs Schreiber posed me in 2005; for $n = 1$, an attempt of an answer was provided in my [14]. I want to thank him for the impetus. I also want to thank Ronnie Brown for having for many years persuaded me to think strictly and cubically. Finally, I want to thank Marco Grandis for useful conversations on cubical and other issues.

1 Preliminaries on cubical sets

The fundamental role of cubical sets (=cubical complexes) have been emphasized through the work of Brown and his collaborators, and by Grandis and his collaborators, and by many others, but is generally not so well documented in the literature. So we start with a brief "lexicon" to establish terminology. This in particular applies to the "symmetry" structures which a cubical set may have, and to the extra "degeneracy"-structure, called "connections" by Brown, Spencer and Higgins. Since the present paper deals with connections in a quite different sense of the word, we shall here use the term "BSH-connections" for the kind of connections which they consider.

We shall mainly follow the scheme of terminology and notations as in [6]. Recall that a cubical set $C_\bullet$ is a family of sets C_n (the "set of *n-cubes* or *n-cells*"; n ranging over the natural numbers), equipped with face- and degeneracy operators:

$$\text{face operators } \partial_i^\alpha : C_n \to C_{n-1} \quad (i = 1, \ldots, n, \alpha = 0, 1)$$

$$\text{degeneracy operators } \epsilon_i : C_{n-1} \to C_n \quad (i = 1, \ldots, n)$$

satisfying the "cubical relation"-equations, as in [2] (1.1) (or [6] (5) for the dual relations), in analogy with the standard relations for the face- and degeneracy operators for simplicial sets.

Just as a simplicial set may carry a richer structure, namely compatible actions of the symmetric groups $\mathfrak{S}_{n+1}$ on the set of n-simplices (in which case it is called a *symmetric* simplicial set by [5]), a cubical set may carry

certain symmetry structures; the symmetries now come in two classes, called respectively, *interchanges* or *transpositions*, and *reversions* in [6].

The *transpositions* which a cubical set may have give rise to an action of $\mathfrak{S}_n$ on the set C_n. This action may be given in terms of operators $\sigma_i : C_n \to C_n$ $(1 \leq i \leq n-1)$ satisfying certain relations and compatibilities with the face- and degeneracy operators, (see [6], equations (27)-(29)). Geometrically, σ_i, applied to an n-cube, interchanges the ith and $(i+1)$st edge emanating from the initial vertex.

The *reversions* are given in terms of operators $\rho_i : C_n \to C_n$ $(1 \leq i \leq n)$ satisfying certain relations, and compatibilities with the face- and degeneracy operators; and also compatibilities with the interchange operators, in case this structure is present, (see [6], equations (56)). Geometrically, ρ_i applied to an n-cube, performs a reflection of the cube in the hyperplane orthogonal to the ith axis.

(In case a cubical set has both transpositions and reversions in a compatible way, the "hyperoctahedral group $(\mathbb{Z}/2)^n \rtimes \mathfrak{S}_n$" acts on the set of n-cubes, see [6] §9.)

In the theory of connections that we develop, the reversions are more important than the transpositions. For, in the cubical groupoids that we consider below, ρ_i is canonically present in terms of the *inversion* of arrows.

Finally, a cubical set may have BSH-connections; these are an extra family of degeneracy operators $\gamma_i : C_{n-1} \to C_n$ $(1 \leq i \leq n-1)$, with compatibility relations with the other structural elements (face-, degeneracy, transposition-, and/or reversion operators). They are denoted Γ_i in [2], and we give some comments and geometry about these in Section 1.6 below. The cubical set of "infinitesimal parallelepipeda" arising from a manifold, which is the main carrier of the theory we present here, has face-, degeneracy-, transposition-, and reversion- operators, but does not have BSH-connections. (The same applies to the cubical set of parallelepipeda in an arbitrary affine space. There is also a notion of parallelepipeda in a non-commutative affine space (pregroup); here, neither BSH-connections nor transpositions are present. This digression, we include in Section 1.5 below.)

1.1 Shells

Let Cub denotes the category of cubical sets, and Cub_n the category of cubical sets "up to dimension n" (such things, we call n-cubical sets). There is an evident truncation functor $tr : Cub_{n+1} \to Cub_n$, forgetting the $n+1$-cells. It is a functor between two presheaf categories, and is induced by a

functor between the respective index-categories, and it therefore has adjoint on both sides. The right adjoint (=the coskeleton), we denote just by $(-)'$ (following [2]), thus if H is an n-cubical set, H' is an $n+1$-cubical set; for $k \leq n$, $H'_k = H_k$, whereas H'_{n+1} consists of "n-shells" $\mathbf{x}$ in H_n, meaning $2(n+1)$-tuples of elements in H_n, whose boundaries match up as if these cells were the $2(n+1)$ faces of an $n+1$-cell in some cubical set. These n-cells then serve as the faces of $\mathbf{x}$ in H'.

If K is an $n+1$-cubical set, there is a front adjunction for the adjointness considered, $\theta : K \to (tr(K))'$; it is the identity in dimensions $\leq n$, and in dimension $n+1$, it associates to an $n+1$-cell x in K its boundary shell $\partial(x)$. (These ideas are from [2].)

All this lifts to cubical sets with reversions and/or transpositions and/or BSH connections. We shall not introduce special notation for each of these enriched notions, but indicate in each case which one of these notions we consider.

1.2 The complex of singular cubes

If M is a manifold, we define the set $S_k(M)$ of singular k-cubes in M to be the set of (smooth) maps $f : R^k \to M$. (The reader may prefer to think of the relevant information of the singular cube f to reside in the restriction of f to the unit cube $I^k \subseteq R^k$, but this unit cube does not play a role in the formalism we present.) The $S_k(M)$ jointly carry structure of a cubical complex $S_{[\bullet]}(M)$: for $\alpha = 0$ or $= 1$, and $i = 1, \ldots, k$, the face map $\partial_i^\alpha : S_k(M) \to S_{k-1}(M)$ consists in precomposing with the affine map $R^{k-1} \to R^k$, $(x_1, \ldots, x_{k-1}) \mapsto (x_1, \ldots, \alpha, \ldots, x_{k-1})$ ($\alpha = 0$ or $= 1$, placed in the ith position), and the degeneracy map $\epsilon_i : S_{k-1}(M) \to S_k(M)$ is induced by projection $R^k \to R^{k-1}$ (omitting ith coordinate). This is also an affine map. There are further canonical affine, hence smooth, maps between the R^ns, giving a richer strucure to $S_{[\bullet]}(M)$, namely transposition of the coordinates, and reversions. We shall also consider these. But one kind of structure will not be present in the purely affine world to be considered next, namely the (BSH-) connections $\gamma_i : S_{k-1}(M) \to S_k(M)$.

If M is an *affine* space, there is a subcomplex $A_{[\bullet]}(M)$ of $S_{[\bullet]}(M)$, with $A_{[n]}(M)$ consisting of the *affine* maps $R^n \to M$ ("*affine* singular n-cubes"). Reversions and transpositions of $S_{[\bullet]}(M)$ restrict to this subcomplex. The information of an affine singular n-cube is of course contained in the images of the 2^n vertices of the unit cube, and they form the vertices of an n-dimensional parallelepipedum in M.

By an *n-simplex* in a set M, we understand an $n+1$-tuple of points in M, called the *vertices* of the simplex. Since R^n is a free affine space on the $n+1$-tuple of points $0, e_1, \ldots, e_n$, it follows that for an *affine* space M, there is a 1-1 correspondence between n-simplices in M, and affine singular n-cubes. We need a notation: to an n-simplex $(x_0, \ldots, x_n)$ in M, we denote the corresponding affine map $R^n \to M$ by $[[x_0, \ldots, x_n]]$; it is given explicitly by the formula

$$[[x_0, \ldots, x_n]](s_1, \ldots, s_n) = (1 - \sum_1^n s_i) \cdot x_0 + \sum_1^n s_i \cdot x_i. \tag{1}$$

Note that the right hand side here is an affine combination, i.e, the sum of the coefficients is 1. Thus, the graded set of simplices in an affine space carries the structure of a (symmetric) simplicial set, as well as of a cubical set (with reversions and transpositions).

Another way to encode the information of an affine singular n-cube $[[x_0, \ldots, x_n]]$ in an affine space M is to consider the 2^n-tuple of points whih are the images of the vertices of the unit cube in R^n under the map $[[x_0, \ldots, x_n]] : R^n \to M$. They form the vertices of an n-dimensional parallelepipedum in M. (This viewpoint will be elaborated algebraically in the Digression below.) The 2^n-tuple of these points we denote $P(x_0; x_1, \ldots, x_n)$; it contains exactly the same information as the simplex $(x_0, \ldots, x_n)$, since the vertices of the latter appear in a canonical way as (some of) the elements of $P(x_0; x_1, \ldots, x_n)$. The 2^n-tuple $P(x_0; x_1, \ldots, x_n)$, we call an *n-dimensional parallelepipedum* in M.

We shall elaborate a little of properties of the bijection between simplices and parallelepipeda in an affine space M, because these properties will also apply for the *infinitesimal* simplices and parallelepipeda which we shall consider. So let M denote an affine space; let $s_{(\bullet)}(M)$ denote the symmetric simplicial set of simplices in it, and $A_{[\bullet]}(M)$ the cubical set, with transpositons and reversions, consisting in the parallelepipeda in M. Then we have for each n the comparison map $p_n : s_{(n)}(M) \to A_{[n]}(M)$ defined by

$$(x_0, x_1, \ldots, x_n) \mapsto P(x_0; x_1, \ldots, x_n).$$

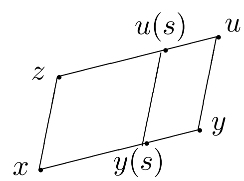
The group $\mathfrak{S}_{n+1}$ acts on $s_{(n)}(M)$, and $\mathfrak{S}_n$ acts as the subgroup of permutations which fix the vertex x_0. Then it is clear that the comparison map $p_n : s_{(n)}(M) \to A_{[n]}(M)$ preserves the action of $\mathfrak{S}_n$. The question of degenerate simplices vs. degenerate cubes is more subtle. A simplex with $x_i = x_{i+1}$ is degenerate in the sense of being a value of a degeneracy operator; whereas a parallelepipedum is a value of a degeneracy operator if

$x_0 = x_i$ for some $i > 0$. The degenerate 2-simplex (x_0, x_1, x_1) goes by p to a parallelogram which is not a value of a cubical degeneracy operator, not even modulo some action of the symmetry group. In fact, for $x_0 \neq x_1$ it has *three* distinct vertices, namely x_0, x_1 and $2x_1 - x_0$.

1.3 Subdivision

We return to the general case where M is a manifold, not necessarily an affine space. We shall consider a structure which $S_{[\bullet]}(M)$ has (we don't here attempt to axiomatize this structure); it is the notion of *subdivision*. Namely, given a scalar $s \in R$, and an index $i = 1, \ldots, n$, we have a "subdivision" operation, which to a singular n-cube $f : R^n \to M$ associates a pair of singular n-cubes f' and f''; they come about by precomposing f by certain affine maps $h_{s,i} : R^n \to R^n$ and $k_{s,i} : R^n \to R^n$, respectively. Here $h_{s,i}$ is the affine map corresponding to the n-simplex $(0, e_1, \ldots, s \cdot e_i, \ldots, e_n)$, and $k_{s,i}$ is the affine map corresponding to the n-simplex $(s \cdot e_i, e_1 + s \cdot e_i, \ldots, e_i, \ldots, e_n + s \cdot e_i)$. Alternatively, $h_{s,i} : R^n \to R^n$ is the the map $\mathrm{id} \times \ldots \times h_s \times \ldots \times \mathrm{id}$, where $h_s : R \to R$ is the map $t \mapsto t \cdot s$; similarly for $k_{s,i}$ with $k_s : R \to R$ the map $t \mapsto s + t \cdot (1 - s)$.

Let us make a picture, and thereby also introduce a notation which we shall need later on. We consider the case of an affine space M (we shall ultimately be interested in the plane R^2), and consider an *affine* singular 2-cube $[[x, y, z]]$, geometrically a parallelogram with vertices x, y, z, u. We consider for each $s \in R$ the point $y(s) := (1 - s)x + sy$, and also the point $u(s) := (1 - s)z + su$.



(2)

For each s, t, $P(x; y(s + t), z)$ subdivides into $P(x; y(s), z)$, $P(y(s); y(s + t), u(s))$. In particular, the displayed parallelogram $P = P(x; y, z)$ subdivides into $P' = P(x; y(s), z)$ and $P'' = P(y(s); y, u(s))$.

If f' and f'' come about from f by such subdivision process (with s and i fixed), we say that f *(i, s)-subdivides into f', f''*. In this case we have the following equations

$$\partial_i^1(f') = \partial_i^0(f'') \tag{3}$$

$$\partial_i^0(f') = \partial_i^0(f) \tag{4}$$

$$\partial_i^1(f'') = \partial_i^1(f). \tag{5}$$

Note that these equations are exactly the book-keeping conditions that one has if f is the i-composite of f' and f'' in a cubical groupoid (in the sense recalled in the next Section). There are also equations relating the j-faces of f with the faces of f' and f'': the formulae are, for $\alpha = 0$, and for $\alpha = 1$,

$$(\partial_j^\alpha(f))' = \partial_j^\alpha(f') \tag{6}$$

$$(\partial_j^\alpha(f))'' = \partial_j^\alpha(f'') \tag{7}$$

and similarly for f''; but for $j < i$, the prime on the left hand side of (6) refers to $h_{s,i-1}$, and the double prime on the left hand side of (7)similarly refers to $k_{s,i-1}$. These equations are all seen simply by considering the respective representing (affine) maps between the relevant R^ms.

These equations can also be formulated

Proposition 1 *Assume f (i,s)-subdivides into f', f''. Then for $j > i$, $\partial_j^\alpha(f)$ (i,s)-subdivides into $\partial_j^\alpha(f')$, $\partial_j^\alpha(f'')$, and for $j < i$, $\partial_j^\alpha(f)$ $(i-1,s)$-subdivides into $\partial_j^\alpha(f')$, $\partial_j^\alpha(f'')$*

1.4 Cubical groupoids

We take the notion of *cubical groupoid* from the fundamental paper [2] by Brown and Higgins. It is what they call ω-groupoids, except that we do not assume that their "connections" Γ_j are part of the structure. The term "cubical groupoid", we have from Grandis. Thus, a cubical groupoid G is a cubical complex, where each G_n is equipped with n (partially defined) compositions $+_i$; the composition $+_i$ makes G_n into the set of arrows of a groupoid, with G_{n-1} as set of objects and with ∂_i^0 and $\partial_i^1 : G_n \to G_{n-1}$ as domain- and codomain-formation, respectively. The degeneracy map $\epsilon_i : G_{n-1} \to G_n$ provides the identity maps for this groupoid structure. Formation of inverse arrows w.r.to $+_i$ is denoted $-_i$ and makes G into a cubical set with reversions. There are several compatibility equations between the structural elements; they may be read off in [2] 1.3, 1.4, and 1.5.

The notion of cubical groupoid truncates in an evident way: thus, by an n-*cubical groupoid*, we understand an n-cubical set G with compositions $+_i$ $(i = 1, \dots, k)$ on each G_k $(k \leq n)$, satisfying the same family of compatibility equations. 1-cubical groupoids are just groupoids; 2-cubical groupoids are double groupoids where the set of horizontal and vertical 1-cells are equal (the "edge-symmetric" double groupoids of Brown and Spencer [3]). The truncation functors $tr : Cub_{n+1} \to Cub_n$ and their right adjoints $(-)'$:

$Cub_n \to Cub_{n+1}$ lift to functors between the categories of n- and $n+1$-cubical groupoids. For tr, this is trivial; for $(-)'$: if G is an n-cubical groupoid, G'_{n+1} consists of *shells* of cells in G_n, and they may be concatenated, or reversed, in each of the $n+1$ directions, using the compositions and reversions of G_n. The exact formulae may be found in [2] §5. In particular, for $n = 1$: if G is an ordinary groupoid, G' is the familiar (edge symmetric) double groupoid whose 2-cells are the (not necessarily commutative) squares in G.

The affine singular cubes in an affine space A form a cubical groupoid; we shall describe its ith composition. Given n-cubes (i.e. n-dimensional parallelepipeda) in A, say P and P' with $\partial_i^1(P) = \partial_i^0(P')$, we describe the n-cube $P \circ_i P'$ as follows. Let $P = P(x_0; x_1, \ldots, x_n)$ and similarly $P' = P(x'_0; x'_1, \ldots, x'_n)$. The compatibility $\partial_i^1(P) = \partial_i^0(P')$ forces, for $j \neq i$, the x'_j to equal certain combinations of the x_ks. Then

$$P \circ_i P' = (x_0; x_1, \ldots, x'_i, \ldots, x_n).$$

Note that if P is (i, s)-subdivided into P', P'', then $P = P' +_i P''$. – Degenerate cubes $s_i(c)$ act as identities for $\circ_i$. We finally describe inversion $-_i$ in the ith direction:

$$-_iP(x_0; x_1, \ldots, x_n) = P(x_i; x_1 - x_0 + x_i, \ldots, x_0, \ldots, x_i - x_0 + x_n)$$

with the x_0 placed in the ith slot.

It is easy to verify that this decribes a cubical groupoid (in the sense of [2]). (It is in fact a cubical equivalence relation: for any two $n-1$-cubes and any $i = 1, \ldots n$, there is at most one n-cube c having the given cubes as ∂_i^0- and ∂_i^1-faces, respectively.)

This cubical groupoid also has transpositions: $\sigma_1(P(x_0; x_1, x_2, \ldots, x_n) = P(x_0; x_2, x_1, \ldots, x_n)$ etc. They will not exist in the non-commutative case which we now consider, as a digression:

1.5 Digression on the non-commutative case

The cubical groupoid structure described above for an affine space A also exists for "non-commutative affine spaces", i.e. for "pregroups" in the sense of [9]; to make the exposition more accessible to the reader not aquainted with pregroups, we consider the special case of a group instead of a pregroup. So let us consider a group G, not necessarily commutative, but we use additive notation. To an $n+1$-tuple $x_0, x_1, \ldots, x_n$ in G, we associate a 2^n-tuple of elements of G in the following way: let $h_1 < h_2 < \ldots < h_k$ be a

k-element subset H of the set $\{1, \ldots, n\}$. If $k \geq 2$, we associate to it the element x_H of G given by the expression

$$x_{h_1} - x_0 + x_{h_2} - x_0 + \ldots - x_0 + x_{h_k};$$

if H is a singleton subset $H = \{h\}$, we put $x_H = x_h$, and if $H = \emptyset$, we put $x_H = x_0$. Note that in the commutative case, all these expressions are affine combinations in G, in fact, the 2^n-tuple descibed is exactly the 2^n-tuple of vertices in the n-dimensional parallelepipedum $P(x_0; x_1, \ldots, x_n)$. Then

$$\partial_i^0(P(x_0; x_1, \ldots, x_n)) = P(x_0; x_1, \ldots, \widehat{x_i}, \ldots, x_n)$$

whereas

$$\partial_i^1(P(x_0; x_1, \ldots, x_n)) = P(x_i; x_{\{1,i\}}, \ldots, \widehat{i}, \ldots, x_{\{i,n\}}).$$

Degeneracies consist in inserting x_0 ($i \geq 1$); the formulae for composition and reversion are the same as described above for affine space. – The transpositions described for the case of affine space A will not work here. For instance, for $n = 2$, there is one transposition operator σ, and $\partial_1^1 \circ \sigma = \partial_2^1$ is one of the required compatibilities. Consider $P(x; y, z)$. Then

$$\partial_1^1(\sigma(P(x; y, z))) = \partial_1^1 P(x; z, y) = P(z; z - x + y)$$

whereas

$$\partial_2^1(P(x; y, z)) = P(z; y - x + z).$$

These may be different, in the non-commutative case.

1.6 Brown-Spencer-Higgins theory

If $C = C_\bullet$ is a cubical set, there is a graded subset $Cr(C)$ of C, where $Cr_n(C) \subset C_n$ for $n \geq 2$ consists of those c such that all faces of c except possibly $\partial_1^0(c)$ are totally degenerate, i.e. come about by applying n degeneracy operators to a 0-cell $\in C_0$. Then ∂_1^0 restricts to a map $\delta : Cr_n(C) \to Cr_{n-1}(C)$. If C is equipped with structure of a cubical groupoid, the graded set $Cr_\bullet(C)$ carries the structure of a *crossed complex* over the groupoid $C_1 \rightrightarrows C_0$ in the sense of [2], in particular, each C_n for $n \geq 2$ is a C_0-indexed family of groups (abelian, if $n \geq 3$), and δ is a group homomorphism, with $\delta \circ \delta = 0$.

We need to recall the theory of connections in the sense of Brown, Spencer, and Higgins. We call them *BSH-connections*. They are extra degeneracies $\gamma_i : C_{n-1} \to C_n$ ($i = 1, \ldots, n-1$) with which a cubical set

$C_\bullet$ may be equipped. For instance, the 2-cubical groupoid G' of squares in an ordinary 1-groupoid G has the extra degeneracy operator γ which to an arrow $g : x \to y$ in G associates the square in G

$$\begin{tikzcd} x \arrow[r, "g"] \arrow[d, "g"'] & y \arrow[d, "\mathrm{id}"] \\ y \arrow[r, "\mathrm{id}"'] & y. \end{tikzcd}$$

If $G = G_\bullet$ is a cubical groupoid with BSH-connections (compatible with the groupoid structures, cf. [2]), there is a retraction ("folding operation", [2]), $\phi : G_n \to Cr_n(G)$ (for all n). The folding operations give a way of expressing what it means for an n-cell in a cubical groupoid to commute:

If G is an n-cubical groupoid with BSH-connections, we are interested in the question of commutativity of $n+1$-cells in the $n+1$-cubical groupoid G'; recall that G' has BSH-connections if G does, so that it has its own folding operations. To say that $\mathbf{x} \in G'_{n+1}$ is commutative is then taken to mean that $\phi(\mathbf{x})$ is the zero element of (one of the groups which constitute) $Cr_{n+1}(G')$. Recall that an element of G'_{n+1} is a shell of n-cells from G. E.g. for $n = 1$, it is a square of arrows from the groupoid G, and the commutativity in the Brown-Higgins sense says that the cyclic composite of the four arrows or their inverses is an identity arrow, and this property is equivalent with what everybody understands as a commutative square in a groupoid.

From Proposition 5.4 in [2], we get

Proposition 2 *Let G be an n-cubical groupoid with BSH-connections; then $Cr_{n+2}(G'') = \{0\}$ (more precisely, it is the G_0-indexed family of 0-groups).*

More information about the relationship between cubical groupoids and crossed complexes is considered in Section 6 below.

2 Infinitesimal parallelepipeda

We now place ourselves in the context of Synthetic Differential Geometry. If M is a manifold, we have the notion of points x, y in M being (1st order) *neighbours*, written $x \sim y$. The neighbour relation $\sim$ is a reflexive symmetric relation. It is not transitive. A k-simplex $(x_0, x_1, \ldots, x_k)$ in M is called an *infinitesimal* simplex if $x_i \sim x_j$ for all $i, j = 0, \ldots, k$.

It was proved in [12] (see also [13]) that if $(x_0, x_1, \ldots, x_k)$ is such an infinitesimal k-simplex, then affine combinations $\sum_0^k t_i \cdot x_i$ may be formed, using a coordinate chart, but independent of the chart; furthermore, any two of these affine combinations are neighbours. Also, any map $M \to N$ preserves the formation of such affine combinations. This implies that the formula (1) describing the map $[[x_0, \ldots, x_k]]$, in case M is an affine space, makes sense even if M is just a manifold, provided $(x_0, \ldots, x_k)$ is an *ininitesimal* simplex. Thus, if $(x_0, \ldots, x_k)$ is an infinitesimal simplex in a manifold M, it defines a singular k-cube $[[x_0, \ldots, x_k]]$ in M. The singular k-cubes in M which arise this way from infinitesimal simplices, we call *infinitesimal k-dimensional parallelepipeda*. The set of these is denoted $M_{[k]}$. The infinitesimal parallelepipeda in M form a subcomplex of the cubical complex $S_{[\bullet]}(M)$, stable under the reversion and transposition operations. Also, a subdivision of an infinitesimal parallelepipedum is an infinitesimal parallelepipedum. – We denote this cubical complex $M_{[\bullet]}$, and its truncation to n dimensions, we denote $M_{[[n]]}$; the inclusion we denote i_n,

$$M_{[[n]]} \xrightarrow{i_n} S_{[[n]]}(M). \tag{8}$$

The best geometric intuition of the cubical complex $M_{[\bullet]}$ is that its k-cells are, or describe, *infinitesimal k-dimensional parallelepipeda* in the following more geometric sense: if $(x_0, \ldots, x_k)$ is an infinitesimal k-simplex in M, it defines a 2^k-tuple of points, namely the images under $[[x_0, \ldots, x_k]] : R^k \to M$ of the 2^k corner points of the unit cube $[0, 1]^k$. To have this intuition playing an active role in our reasoning, we write $P(x_0; x_1, \ldots, x_k)$ for this $2k$-tuple, even though it contains exactly the same information as the infinitesimal k-simplex $(x_0, \ldots, x_k)$, or as the singular k-cube $[[x_0, \ldots, x_k]]$. But note that the "unit interval" $I = [0, 1]$, as a point set $\{s \in R \mid 0 \leq s \leq 1\}$, and similarly, the "unit cube" I^k, do not play any formal role in our treatment. (They can be incorporated in a more refined theory, where one takes a preordering $\leq$ of the line R into account. If we let $\leq$ be the "chaotic" preordering ($x \leq y$ for all $x, y \in R$), then $R = [0, 1]$ and $R^n = [0, 1]^n$, justifying the terminology that $R^n \to M$ is a singular *cube* in M.)

Note that the described "parallelepipedum-formation" establishes for each k a bijection p_k between the set $M_{(k)}$ of infinitesimal k-simplices in M, and the set $M_{[k]}$ of infinitesimal k-dimensional parallelepipeda in M. So we have the same phenomenon as for affine space: a graded set which carries structure of (symmetric) simplicial complex, as well as of cubical complex (with reversions and transpositions, and even with subdivisions).

The reader should be warned that the infinitesimal parallelograms which

arise in this way are more special than the parallelograms through which a notion of affine connection may be codified, as in [10] (1.4); there, x, y, z is supposed to satisfy $x \sim y$ and $x \sim z$, but not $y \sim z$; without the assumption of $y \sim z$, the formation of the parallelogram spanned by x, y, z is an added *structure*, not canonical.

3 Higher connections, and their curvature

It is desirable that certain mathematical structures can be encoded as *morphisms* in a suitable category. In the context of SDG, a certain general notion of *connection* can been so encoded: recall [11] that if G is a groupoid with $G_0 = M$ a manifold, then a *connection* in G may be construed as a morphism of reflexive symmetric graphs $M_{(1)} \to G$ over M.

We now describe how a certain notion of *higher* connection arises in a similar way as a morphism in a suitable category. Note that a reflexive symmetric graph is the same as a 1-cubical set with reversions. Recall that a cubical groupoid has an underlying cubical set with reversions.

Definition 3 *Let $G_\bullet$ be a cubical groupoid with object set M. An ω-*connection *in G is a map ∇ of cubical sets with reversion $M_{[\bullet]} \to G$ over M.*

So for each infinitesimal k-dimensional parallelepipedum P in M, we have a k-cell $\nabla_k(P)$ in G, in a way which is compatible with face-, degeneracy-, and reversion-operators.

We have truncated versions, for any finite k:

Definition 4 *Let G be a k-cubical groupoid with object set M. A k-*connection *is a map ∇ of cubical sets with reversion $M_{[[k]]} \to G$ (preserving M) (where $M_{[[k]]}$ denotes the k-truncation of $M_{[\bullet]}$).*

Since $M_{[[k]]}$ is the k-truncation of $M_{[[k+1]]}$, we get from the adjointness $sk \dashv (-)'$ a bijective correspondence between the following two kinds of entries

$$\begin{array}{ll} k\text{-connections} & \nabla : M_{[[k]]} \to G \\ \hline k+1\text{-connections} & \widehat{\nabla} : M_{[[k+1]]} \to G'. \end{array}$$

In dimensions $\leq k$, ∇ and $\widehat{\nabla}$ agree; in dimension $k+1$, $\widehat{\nabla}_{k+1} : M_{[k+1]} \to G'_{k+1}$ is what we shall call the *formal curvature* of ∇. Or, we may call $\widehat{\nabla}$ itself the formal curvature of ∇. With this twist of terminology, the formal curvature

of a k-connection is a $k+1$-connection, and this process may then be iterated; this is exploited in the formulation if a "Formal Bianchi Identity" below, Theorem 6.

(The view that the curvature of a k-connection is a flat $k+1$-connection has also been developed by U. Schreiber [18], in the context of connections as "transport along paths".)

Thus, for $k = 1$, this means that for $x \sim y$ in M, $\nabla(P(x;y)) = \nabla(x,y)$ is an arrow $x \to y$ in G, and for an infinitesimal parallelogram $P = P(x;y,z)$ with fourth vertex u, $\widehat{\nabla}(P)$ is the square (shell) in G

$$\begin{tikzcd} z \arrow[r, "\nabla(z,u)"] & u \\ x \arrow[u, "\nabla(x,z)"] \arrow[r, "\nabla(x,y)"'] & y \arrow[u, "\nabla(y,u)"'] \end{tikzcd} \tag{9}$$

and this diagram is only commutative for all such P if ∇ is curvature free (flat), in the sense of [11]. Thus $\widehat{\nabla}$ in this case encodes the information of the curvature of ∇, which is the reason for the name "formal curvature". In this case ($k = 1$), we may get the standard [1] (synthetic) curvature as the function R which assigns to P the arrow $u \to u$ given by taking the cyclic composite of the arrows in the displayed diagram,

$$R(P) = \nabla(u,y).\nabla(y,x).\nabla(x,z).\nabla(z,u). \tag{10}$$

For higher k, one does not have an analogous construction without a further structure on the cubical groupoid $G_\bullet$; the structure needed is BSH-connections. Any 1-groupoid carries canonical such (because it is here a void concept); if G is an n-groupoid carrying BSH-connections, then $\widehat{G}$ carries them, too, in a canonical way, see [2].

Recall from [2] that for a cubical groupoid G with BSH connections, we have "folding operations" $\phi : G_k \to Cr_k(G)$; $Cr_k(G)$ being, for $k \geq 2$, a certain family of groups indexed by G_0. We can use these to reorganize the notion of curvature: Let G be a k-cubical groupoid with $G_0 = M$ a manifold, and let ∇ be a k-connection in it. We describe the "real" curvature of it

[1] it is not really "standard"; but for the corresponding simplicial notion of connections, as in [11], curvature was defined in terms of a similar cyclic composite $R(P) := \nabla(x,y).\nabla(y,z)\nabla(z,x)$. A four-fold cyclic composite for defining curvature cubically has also been considered by [17].

in terms of the formal curvature $\widehat{\widehat{\nabla}} : M_{[[k+1]]} \to G'$ previously described, by composing the formal curvature with the folding:

Definition 5 *The curvature R_∇ of ∇ is the composite map*

$$M_{[k+1]} \xrightarrow{\widehat{\widehat{\nabla}}_{k+1}} G'_{k+1} \xrightarrow{\phi} Cr_{k+1}(G').$$

The k-connection ∇ is called flat *if its curvature is 0.*

The reader may prefer to further compose with the map $\delta : Cr_{k+1}(G') \to Cr_k(G') = Cr_k(G)$, which is monic, so does not lose any information.

For $k = 1$: consider a 1-connection in an ordinary groupoid G. Then G' has a canonical BSH-connection, as described in Section 1.6. The corresponding folding operation $\phi : G'_2 \to Cr_2(G')$ associates to a square in G the cyclic composite of its four constituents, as in (10). ($Cr_2(G')$ is the M-indexed family of the vertex groups of G). Thus in particular, the connection is flat iff for any infinitesimal parallelogram, the square exhibited in (9) commutes, which is the (cubical rendering of the simplicial) flatness notion as described in [11].

Consider again is a k-cubical groupoid G with BSH-connections. Then G' is a k+1-cubical groupoid with BSH-connections, and G'' is a k+2-cubical groupoid with BSH-connections.

We have

Theorem 6 (Formal Bianchi Identity) *If ∇ is a k-connection in a k-cubical groupoid G with BSH-connections, then the $k+1$-connection $\widehat{\widehat{\nabla}}$ in the $k+1$-cubical groupoid G' (the formal curvature of ∇) is flat.*

Proof. This is an immediate consequence of the fact that $Cr_{k+2}(G'')$ consists of trivial groups $\{0\}$, by Proposition 2.

We would like to describe what this means for $k = 1$, entirely in terms of G, without reference to the derived structures G', G'', not to speak of the folding operations ϕ, which are rather complicated. It is, for a 3-dimensional parallelepipedum P in M, entirely a question of the values of ∇ on the 12 1-dimensional edges of P; they form the 12 edges of a cube-shaped diagram $\mathbf{x}$ in G. Now, Proposition 2 for $n = 1$ can be read as an identity which holds for all such cube-shaped diagrams in all groupoids. The Homotopy Addition Lemma ([2] Lemma 7.1) provides simpler expressions for some of the values

of the folding operations ϕ; for the present case, the value of ϕ given in loc.cit. (Lemma 7.1, case $n = 2$): it is the right hand side of the following 30-letter identity for 12 elements and their inverses in a groupoid; this identity holds whenever the book-keeping conditions make the expression meaningful. We need to give names to the 12 elements involved i.e. to the edges of the cube-shaped diagram. We number the vertices of the cube by the integers 0, $\ldots$,7; the vertex corresponding to an integer $0, \ldots, 7$ is then the one whose coordinates in the cube are the digits of the integer when written (with three digits) in binary notation, thus e.g. 3 corresponds to the vertex $(0, 1, 1)$ since 3 in binary notation is 011. Then the arrows (edges) are denoted by their domain and codomain; e.g. 37 denotes the arrow from vertex 3 to vertex 7, and 73 denotes its inverse. Note that for instance the symbol 06 does not correspond to an edge of the cube: the twelve edges in question are 01,02,04,13,15,23,26,37,45,46,57,67. The identity then is the following; the parentheses can be ignored, but are placed there so that one can see each of the six curvatures, corresponding to the six faces of the cube. The remaining six arrows perform the needed three conjugations which bring those three curvatures that live at other vertices than 7 over to vertex 7:

$$\begin{aligned}\mathrm{id}_7 = \; &(76\ 64\ 45\ 57)\ 75\ (54\ 40\ 01\ 15)\ 57\ (75\ 51\ 13\ 37) \\ &\qquad 73\ (31\ 10\ 02\ 23)\ 37\ (73\ 32\ 26\ 67)\ 76\ (62\ 20\ 04\ 46)\ 67\end{aligned} \tag{11}$$

It is analogous to the Ph. Hall's “14 letter identity” used in [11] for a simplicial-combinatorial proof of the Bianchi identity. A variant of the above identity from [2] was found independently bi Nishimura [17], who also used it to give a proof of the Bianchi identity for connections in groupoids.

For future reference, we rewrite the formula (11) in a (hopefully self-explanatory) notation, with R denoting curvature, and exponents denoting conjugation:

$$\mathrm{id}_7 = R(456).R(041)^{57}.R(153).R(012)^{37}.R(236).R(024)^{67}; \tag{12}$$

here, for instance (024) denotes the square with vertices 0,2,4 and 6, and similarly for the other five R-expressions.

4 Holonomy; Stokes' Theorem

Let M be a manifold, and let G be an n-cubical groupoid with $G_0 = M$, so that we may talk about n-connections in G. We intend to look for “integral” aspects of such connections (i.e. *global* aspects, rather than infinitesimal).

Recall that we have the cubical set $S_{[\bullet]}(M)$ of singular cubes in M, and we have its n-truncation $S_{[[n]]}(M)$. Its set of 0-cells may be identified with M itself. We may therefore consider morphisms of n-cubical sets

$$\Sigma : S_{[[n]]}(M) \to G$$

with $\Sigma_0 = \mathrm{id}_M$. Recall the notion of subdivisions of singular cubes, cf. Section 1.3.

Definition 7 *An* abstract holonomy *in G is a morphism Σ of n-cubical sets $S_{[[n]]}(M) \to G$ which furthermore has the following subdivision property: if an $f \in S_k(M)$ subdivides in the ith direction into f' and f'', then*

$$\Sigma(f) = \Sigma(f') +_i \Sigma(f'').$$

The subdivision property implies that an abstract holonomy Σ preserves the reversion structure present in both $S_{[[n]]}$ and in G.

Recall that the n-cubical set $S_{[[n]]}(M)$ also carries the structure of transpositions τ_i; suppose that the n-cubical groupoid is equipped with such structure as well. Then we may ask that an abstract holonomy preserves the transpositions. We then have a stronger notion:

Definition 8 *An abstract holonomy Σ, as in the previous definition, is called* alternating *if it preserves the transposition structure.*

In Section 6 below, we shall prove that a differential n-form ω on M gives rise to an n-connection (in a certain n-cubical groupoid), and that integration of ω over singular n-cubes in M is an alternating holonomy.

The n-cubical set $M_{[[n]]}$ of infinitesimal parallelepipeda in M of dimension $\leq n$ embeds into the n-cubical set $S_{[[n]]}(M)$, (8): $i_n : M_{[[n]]} \to S_{[[n]]}(M)$. It is clear that the truncation functor $tr : Cub_{n+1} \to Cub_n$ takes i_{n+1} to i_n. Therefore, the following diagram is commutative, by virtue of naturality of the hom-set bijections for the adjointness $tr \dashv (-)'$.

$$\begin{tikzcd}
\hom(S_{[[n+1]]}(M), G') \arrow[r, "i^*_{n+1}"] & \hom(M_{[[n+1]]}, G') \\
\hom(S_{[[n]]}(M), G) \arrow[r, "i^*_n"'] \arrow[u, "\widehat{\ }", "\cong"'] & \hom(M_{[[n]]}, G), \arrow[u, "\cong", "\widehat{\ }"']
\end{tikzcd} \tag{13}$$

where the vertical maps are the hom set bijections $\widehat{(-)}$.

The homs in this diagram are the hom-set formation for the category Cub_{n+1} in the upper line, and Cub_n in the lower line; However, we have a similar diagram if hom denotes the hom sets for the categories of cubical sets with reversion, or cubical sets with reversion and transpositions (provided, in the latter case, that G is equipped with transpositions). For, the adjointness lifts to these more structured categories, and the inclusion $i_n : M_{[[n]]} \to S_{[[n]]}(M)$ preserves this structure (canonically present in the n-cubical sets $M_{[[n]]}$ and $S_{[[n]]}(M)$); and similarly for i_{n+1}.

If $\nabla : M_{[[n]]} \to G$ is an n-connection, one may ask whether an abstract holonomy $\Sigma : S_{[[n]]}(M) \to G$ extends ∇, in other words, one may ask whether

$$\nabla(P(x_0; x_1, \ldots, x_k)) = \Sigma([[x_0, \ldots, x_k]])$$

for any infinitesimal k-simplex $(x_0, \ldots, x_k)$ in M ($k \leq n$). Since an abstract holonomy by definition has the subdivision property, a necessary condition that ∇ extends into an abstract holonomy is that ∇ itself has the subdivision property; I don't know whether this in automatic in general, but I know cases where it is, see Section 7 below.

We say that ∇ is *integrable* if there is a *unique* abstract holonomy Σ extending ∇; in this case, the Σ deserves the name *the holonomy of* ∇ and is denoted $\int \nabla$; its value on a singular k-cube f is denoted by $\int_f \nabla$.

If the n-cubical groupoid G is furthermore equipped with transposition structure, we have a variant of the notion of integrability: namely we say that ∇ is *a-integrable* if there is a unique *alternating* holonomy Σ extending ∇, in which case again this Σ is denoted $\int \nabla$.

Examples of n-cubical groupoids G, in which every connection is a-integrable, are given in Section 7 below; examples of 1-cubical groupoids (=groupoids), where every connection is integrable, were given in [14].

Let G be an n-cubical groupoid, and $\Sigma : S_{[[k]]}(M) \to G$ a morphism of n-cubical sets.

Lemma 9 *If Σ has the subdivision property, then so does $\widehat{\Sigma} : S_{[[k+1]]}(M) \to G'$.*

Proof. It suffices to check dimension $n+1$. Let $f \in S_{[n+1]}(M)$ (i, s)-subdivide into f', f''. We must prove

$$\widehat{\Sigma}(f) = \widehat{\Sigma}(f') +_i \widehat{\Sigma}(f').$$

As elements of G'_{n+1}, both sides are determined by their faces, so it suffices to see for each $j = 1, \ldots, n+1$ and $\alpha = 0, 1$ that

$$\partial_j^\alpha(\widehat{\Sigma}(f)) = \partial_j^\alpha\big(\widehat{\Sigma}(f') +_i \widehat{\Sigma}(f'')\big).$$

There are three cases, $j < i$, $j > i$ and $j = i$. Consider the first case.

We have, by construction of $\widehat{\Sigma}$,

$$\partial_j^\alpha(\widehat{\Sigma}(f)) = \Sigma\partial_j^\alpha(f).$$

Now by Proposition 1, $\partial_j^\alpha(f)$ $(i-1, s)$-subdivides into $\partial_j^\alpha(f')$ and $\partial_j^\alpha(f'')$, and since Σ has the subdivision property, we get for the right hand side of the above equation the expression

$$\Sigma(\partial_j^\alpha(f'))+_{i-1}\Sigma(\partial_j^\alpha(f'')) = \partial_j^\alpha(\widehat{\Sigma}(f'))+_{i-1}\partial_j^\alpha(\widehat{\Sigma}(f'')) = \partial_j^\alpha\big((\widehat{\Sigma}(f')+_i\widehat{\Sigma}(f'')\big),$$

the last equality sign by one of the rules for cubical groupoids ([2] (1.3.i)).

This proves the desired equation for the case $j < i$. The case $j > i$ is similar, except that i does not change to $i-1$ during the process. Finally, consider $i = j$. It divides into two cases, $\alpha = 0$ and $\alpha = 1$. For $\alpha = 0$, we have

$$\partial_i^0\big(\widehat{\Sigma}(f') +_i \widehat{\Sigma}(f'')\big) = \partial_i^0\big(\widehat{\Sigma}(f')\big) = \Sigma\big(\partial_i^0(f')\big) = \Sigma\big(\partial_i^0(f)\big),$$

using (4) twice. But this equals $\partial_i^0(\widehat{\Sigma}(f))$, and proves the desired equality. The case $\alpha = 1$ is similar, using (5).

Consider now an n-cubical groupoid G, with $G_0 = M$ a manifold.

Theorem 10 (Formal Stokes' Theorem) *Consider an n-connection $\nabla : M_{[[n]]} \to G$ in G; if it is integrable, then so is its formal curvature $\widehat{\nabla} : M_{[[n+1]]} \to G'$; and*

$$\int \widehat{\nabla} = (\int \nabla)\widehat{}.$$

If G is provided with transpositions, the same holds if the word "integrable" is replaced by "a-integrable".

Proof. Let $\nabla : M_{[[n]]} \to G$ be an integrable connection, with holonomy $\int \nabla : S_{[[n]]}(M) \to G$. To say that $\int \nabla$ extends ∇ is to say that $i_n^*(\int \nabla) = \nabla$, and from the commutativity of the diagram (13), it follows that $i_{n+1}^*(\int \nabla)\widehat{}= \widehat{\nabla}$, which is to say that $(\int \nabla)\widehat{}$ extends $\widehat{\nabla}$. Also, since $\int \nabla$ has the subdivision property, it follows from Lemma 9 that so does $(\int \nabla)\widehat{}$. This proves the existence of an abstract holonomy extending $\widehat{\nabla}$. The uniqueness is essentially trivial: elements in $\hom(S_{[[n+1]]}, G')$ $(i = 1, 2)$ correspond under the hom-set bijection displayed in (13) to $\Sigma_i \in \hom(S_{[[n]]}(M), G)$, i.e. they are of the form $\widehat{\Sigma}$. Given thus $\widehat{\Sigma}_i \in \hom(S_{[[n+1]]}, G')$ $(i = 1, 2)$, if they both have

the subdivision property, then so do Σ_1 and Σ_2; and if the $\widehat{\Sigma}_i$ both restrict to $\widehat{\nabla}$, then both Σ_is restrict to ∇, and hence $\Sigma_1 = \Sigma_2$, by the uniqueness assumption (integrability of ∇). But then also $\widehat{\Sigma}_1 = \widehat{\Sigma}_2$.

The proof of the assertion about a-integrability is similar, using that the diagram (13) also may be read with hom denoting the category of cubical sets with transpositions and reversions. – This proves the Theorem.

The reason for calling it a "Stokes' Theorem" is that, for given $f \in S_{n+1}(M)$, $\int_f \widehat{\nabla}$ may be read as the integral of the coboundary (=curvature) $\widehat{\nabla}$ of ∇ over the $n+1$-cube f, whereas $(\widehat{\int \nabla})(f)$ may be read as the integral of ∇ over the boundary shell of f, recalling that the values of any $\widehat{h}$ on an $n+1$-cube is obtained in terms of the values of h on the boundary shell of the cube. This will be elaborated in Section 7, where we shall consider Stokes' Theorem for differential forms.

5 Cubical-combinatorial differential forms

We shall give a short exposition of combinatorial differential forms, both simplicially and cubically. A fuller account may be found in [15].

Let M be a manifold. Recall from Section 2 that we have for each n a bijection p_n from the set $M_{(n)}$ of infinitesimal n-simplices in M to the set $M_{[n]}$ of infinitesimal n-dimensional parallelepipeda in M. Thus there is also a bijective correspondence p_n^* between functions $M_{[n]} \to R$ and functions $M_{(n)} \to R$. One of the equivalent ways to say that such a function $M_{(n)} \to R$ is a *combinatorial differential n-form* (or simplicial-combinatorial differential n-form on M) is that it takes value 0 on any $(x_0, \ldots, x_n)$ where $x_0 = x_i$ for some $i > 0$. Simplices of this form are precisely those that under p_n correspond to values of the cubical degeneracy operators (such we call *degenerate* cubes). Therefore we put

Definition 11 *A* cubical-combinatorial differential n-form *(briefly, a cubical n-form) on M is a function $M_{[n]} \to R$ which take degenerate n-cubes to 0.*

(When we in the following say "n-form" without further decoration, we mean "cubical-combinatorial differential n-form".)

So the bijection p_n^* restricts to a bijection between simplicial-combinatorial n-forms and cubical-combinatorial n-forms.

Now both $M_{(\bullet)}$ and $M_{[\bullet]}$ are *symmetric* simplicial (resp. cubical) sets: the symmetric group $\mathfrak{S}_{n+1}$ acts on $M_{(n)}$, and $\mathfrak{S}_n$ acts on $M_{[n]}$. Also $\mathfrak{S}_n$ acts

on $M_{(n)}$, as those permutations in $\mathfrak{S}_{n+1}$ which keep the first vertex fixed. Simplicial-combinatorial n-forms are known to be *alternating*, in the sense that $\omega(\tau(x)) = \mathrm{sign}(\tau) \cdot \omega(x)$ for any $\tau \in \mathfrak{S}_{n+1}$, $x \in M_{(n)}$.

As we observed in Section 1.2 for the case of affine simplices and parallelepipeda, p_n preserves the action of $\mathfrak{S}_n$; this also holds for infinitesimal simplices/ parallelepipeda; so it immediately follows that cubical-combinatorial n-forms are likewise $\mathfrak{S}_n$-alternating.

These assertions, however, do not exhaust the symmetry properties of neither the simplicial n-forms, nor the cubical n-forms; simplicial n-forms are not only alternating w.r.to the action of $\mathfrak{S}_n$, but with respect to the action of $\mathfrak{S}_{n+1}$ on $M_{(n)}$; and cubical n-forms are alternating w.r.to the "reversion" structure which the set $M_{[n]}$ carries. For this latter assertion, let us be more explicit. By a coordinate argument, one may prove

Proposition 12 *Let ω be a cubical-combinatorial k-form on M. Then for any infinitesimal parallelepipedum P, $\omega(-_iP) = -\omega(P)$, where $-_iP$ is the reversion of P in the ith direction $(i = 1, \ldots, k)$.*

We refer to [15] for a proof. It is essentially the same as the proof that gives the "extra" symmetry property ($\mathfrak{S}_{n+1}$-alternating property) for simplicial n-forms.

On R^n, there is a canonical n-form, the "volume form" Vol. For an infinitesimal parallelepipedum $P = P(x_0; x_1, \ldots, x_n)$, it is given by the formula

$$\mathrm{Vol}(P) := \det(x_1 - x_0, \ldots, x_n - x_0),$$

the determinant of the $n \times n$ matrix whose columns are $x_i - x_0$. (The form Vol may also be defined as $dt_1 \wedge \ldots \wedge dt_n$, with the wedge product as described in [15].)

We shall need the following result, see [15]:

Proposition 13 *For every cubical-combinatorial n-form θ on R^n, there exists a unique function $\widehat{\theta} : R^n \to R$ such that for any infinitesimal parallelepipedum $P = P(x_0; x_1, \ldots, x_n)$*

$$\theta(P) = \widehat{\theta}(x_0) \cdot \det(x_1 - x_0, \ldots, x_n - x_0),$$

i.e. such that $\theta = \widehat{\theta} \cdot \mathrm{Vol}$.

Since arbitrary functions $f : N \to M$ between manifolds take infinitesimal parallelepipeda to infinitesimal parallelepipeda we have, just as for

simplicial-combinatorial forms, that the set of cubical n-forms on manifolds depends contravariantly on the manifold; thus, if θ is a cubical n-form on M, we get a cubical n-form $f^*(\theta)$ on N, with

$$f^*(\theta)\big(P(y_0;y_1,\ldots,y_n)\big) := \theta\big(P(f(y_0);f(y_1),\ldots,f(y_n))\big),$$

where the y_is are mutual neighbours in N. We are going to use this for the case where f is a map $R^n \to M$ given by some infinitesimal n-parallelepipedum in M; namely, we have the following Proposition from [15]. It is proved by a Taylor expansion argument, together with the product rule for determinants:

Proposition 14 *Let ω be a cubical n-form on a manifold M, and let $P = P(x_0;x_1,\ldots,x_n)$ be an infinitesimal n-dimensional parallelepipedum in M . Then we have the following equality of n-forms on R^n:*

$$[[x_0,\ldots,x_n]]^*(\omega) = \omega(P)\cdot \mathrm{Vol}\,.$$

Note that the right hand side is a "constant" n-form on R^n in the sense that the function $\widehat{\theta}$ corresponding, by Proposition 13, to it, is a constant function (constant with value $\omega(P)$).

In Section 7, we shall consider the question of integrating n-forms along singular n-cubes.

For a simplicial-combinatorial n-form ω, one constructs a simplicial-combinatorial $n+1$-form $d\omega$, by the usual formula for coboundary of simplicial cochains; see [11] §4. For cubical-combinatorial n-forms, one also has a coboundary operator, by the ususal formula for coboundary of cubical cochains (as in [7] §8.3, say). These two coboundaries match modulo a factor $n+1$: denoting the coboundaries respecively by d_s and d_c (for "simplicial" and "cubical", respectively), the formula is

$$d_s(\omega) = \frac{1}{n+1} d_c(\omega),$$

(where we omit the bijections p_n^* and p_{n+1}^* from notation). The cubical formula can be seen, when working in a coordinatized situation, to give the standard formula for exterior derivative of "classical" differential forms (via a well known correspondence between combinatorial and classical differential forms, see [15]).

We have have $d(f^*(\theta)) = f^*(d\theta)$.

6 Connections as generalized forms

We shall see how the notion of "connection in an n-cubical groupoid" contains as a special case the notion of differential n-form. This comes about by considering certain "constant" n-cubical groupoids. (For $n = 1$, this was described in [14]; here the "value group" A need not be assumed commutative.)

Let M be any set (manifold), and $(A, +)$ any abelian group (ultimately, we shall be interested in the case where $A = R$, the number line). For $n \geq 1$, we may consider the n-cubical groupoid $M_n(A)$ whose k-cells for $k < n$ form the set M^{2^k}; and the set of n-cells is taken to be $M^{2^n} \times A$. (It is the cartesian product in the category of n-cubical groupoids of the codiscrete n-cubical groupoid on M, on the one side, and, on the other side, the n-cubical groupoid with A as set of n-cells, a one-point set as set of k-cells, for $k < n$, and where all n compositions $+_i$ of n-cells are taken to be $+$.

The n-cubical groupoid $M_n(A)$ has BSH connections (using $0 \in A$).

The category of n-cubical groupoids with BSH-connections is, by [2], equivalent to the category of crossed n-comlexes over groupoids, as described in loc.cit.; such a thing consists in an "ordinary" groupoid $G \rightrightarrows M$, and for each $n \geq 2$, an M-indexed family of (n-truncated) chain complexes $C_n \xrightarrow{\delta} C_{n-1} \to \ldots C_3 \xrightarrow{\delta} C_2$ (where C_2 is the family of vertex groups of $G \rightrightarrows M$; C_2 need not be abelian; the groupoid $G \rightrightarrows M$ acts on these chain complexes in a compatible way. We shall not need a more complete description, but just describe the crossed n-complex (respectively crossed $n+1$-complex) corresponding to $M_n(A)$ (respectively to $(M_n(A))'$). In both cases, the groupoid is the codiscrete $M \times M \rightrightarrows M$, and it acts trivially. For each $m \in M$, the corresponding chain complexes are, respectively, A concentrated in dimension n (and 0s elsewhere); and A in dimension $n + 1$ and also in dimension n, with the identity map as δ, and 0s elsewhere.

For any n-cubical groupoid G, let us denote the corresponding crossed complex by $Cr(G)$. It agrees with G in dimension 0 and 1. Its k-dimensional part $Cr_k(G)$ ($k \geq 2$) is a subset of G_k, consisting of those k-cells all of whose faces except ∂_1^0 are "totally degenerate", i.e. comes about from a 0-cell by applying k degeneracies. Then (the restriction of) ∂_1^0 will serve as $\delta : Cr_k(G) \to Cr_{k-1}(G)$. According to [2], there is for each k a retraction $\phi : G_k \to Cr_k(G)$; it is a somewhat complicated "folding operation". However, for $M_n(A)$ and $(M_n(A))'$, the folding operations are easy to describe. In dimensions $k < n$, it is the map $l : M^{2^k} \to M$ which pick out the last vertex

of a 2^k-tuple. In dimension n, it is the map $l \times \mathrm{id}_A : M^{2^n} \times A \to M \times A$ (these descriptions apply both to $M_n(A)$ and to $(M_n(A))')$, which agree in dimensions $\leq n$). Finally, for $(M_n(A))'$, we need to describe ϕ in dimension $n+1$. It follows from the "Homotopy Addition Lemma", [2] Lemma 7.1, that ϕ here may be described as follows. An element in $(M_n(A))'_{n+1}$ consists of a 2^{n+1}-tuple $\mathbf{x}$ of elements from M, together with a $2(n+1)$ tuple of elements from A, one for each face operator ∂_i^α ($\alpha = 0, 1$, $i = 1, \ldots, n+1$). Denoting the element $\in A$ corresponding to ∂_i^α by $a_i^\alpha \in A$, ϕ gives the value

$$\big(l(x), \sum_{i=1}^{n+1} (-1)^i \{a_i^1 - a_i^0\}\big). \tag{14}$$

Note that in the quoted Lemma, it is $\delta \circ \phi$ rather than ϕ which is described, but in our case δ is an identity map. Also note that all the groupoid actions appearing in the formula, can be ignored; the action is trivial. Finally note that the $l(x) \in M$ components in the formulae for ϕ contain no information, all information resides in the component $\in A$.

We now consider the case $A = R$, for concreteness; for the case $n = 1$, non-commutative Lie groups may be used for A, see [14]). For $n = 2$, see the Remark at the end of the present Section. Then we have

Proposition 15 *There is a bijective correspondence between connections ∇ in the n-cubical groupoid $M_n(R)$, and differential (R-valued) n-forms ω on M. To ∇, the corresponding ω is given as the composite*

$$M_{[n]} \xrightarrow{\nabla_n} M^{2^n} \times R \xrightarrow{\text{proj}} R.$$

Also, connections in $M_n(A)$ are automatically alternating.

Proof. To see that the exhibited map is a (cubical-combinatorial) n-form, it suffices to see that it takes value 0 on degenerate n-cells; but ∇ takes degenerate n-cells into identity n-arrows in the groupoid, and identity arrows have their A-component equal 0. – On the other hand, given an n-form ω, we define $\nabla : M_{[[n]]} \to M_n(R)$ as follows: in dimensions $k < n$, it is just the inclusion map $i : M_{[k]} \to M^{2^k}$ associating to an infinitesimal k-dimensional parallelepipedum its 2^k-tuple of vertices; and in dimension n, it associates to an infinitesimal n-dimensional parallelepipedum P the n-cell given by $(i(P)), \omega(P)) \in M^{2^n} \times R$. This clearly describes a morphism of cubical sets with reversions It is clear that the correspondence is bijective. Since differential n-forms ω are known to be automatically alternating, the last assertion of the Proposition follows.

With ∇ and ω as in this Proposition, we shall next relate the formal curvature $\widehat{\nabla}$ of ∇ with the exterior derivative of the ω.

Proposition 16 *With ∇ and ω as in the previous Proposition, the composite*

$$M_{[n+1]} \xrightarrow{\widehat{\nabla}_{n+1}} (M_n(R))'_{n+1} \xrightarrow{\phi} M \times R \xrightarrow{\text{proj}} R$$

equals $d\omega$.

Proof. Recall that the cubical-combinatorial $n+1$-form $d\omega$ is given by the standard cubical coboundary of ω, thus for P an infinitesimal $n+1$-dimensional parallelepipedum

$$d\omega(P) = \sum_{i=1}^{n+1} (-1)^i \{\omega(\partial_i^1(P)) - \omega(\partial_i^0(P))\}.$$

Now the result follows by comparison with (14).

Remark. We finish by a remark on 2-forms with non-commutative values. Recall that a crossed complex need not be commutative in dimension 2. By the equivalence between cubical groupoids with BSH connections and crossed complexes, we have the possibility of deriving a notion of 2-form with non-commutative values, and its coboundary, by passing via the curvature of the corresponding 2-connection. We shall just sketch this. (The reader may want to specialize to the case where the crossed complex is constant, as we did when we viewed R-valued n-forms as n-connections.)

So consider a crossed module (= 2-crossed complex) corresponding to a 2-cubical groupoid G with BSH-connections. The 0-dimensional part is supposed to be a manifold M. To fix notation, we exhibit this crossed module $Cr(G)$:

$$C_2 \xrightarrow{\delta} G_1 \rightrightarrows M.$$

A 2-connection ∇ in G gives, via the folding operation ϕ, rise to the following data with values in the crossed module $Cr(G)$:

1) A 1-connection ∇_1 in the 1-groupoid $G_1 \rightrightarrows M$;

2) A function ω on $M_{[2]}$ with values in the group bundle C_2 over M satisfying, for all infinitesimal parallelograms $P(x_0; x_1, x_2)$

$$\delta(\omega(P(x_0; x_1, x_2)) = R(P(x_0; x_1, x_2)).$$

Here, ω is a group-bundle-valued 2-form, in the sense that $\omega(P(x_0; x_1, x_2)) \in C_2(x_3)$, (= the group corresponding to the fourth vertex $x_3 \in M$ of the

parallelogram $P(x_0; x_1, x_2)$ [2]), and so that the value is 0 if the parallelogram is degenerate (we are using additive notation in the groups that make up C_2). Also, R denotes the (real) curvature of the connection ∇_1, arising from the formal curvature $\widehat{\nabla_1}$ by the folding.

Now the formal curvature $\widehat{\nabla}$ of the 2-connection is a 3-connection in G', which is a 3-cubical groupoid with BSH-connections. To this 3-cubical groupoid corresponds the 3-crossed complex

$$\mathrm{Ker}(\delta) \hookrightarrow C_2 \xrightarrow{\delta} G \rightrightarrows M.$$

(Here, $\mathrm{Ker}(\delta)$ is contained in the center of C_2, and it is in particular commutative.) So the formal curvature $\widehat{\nabla}$ gives by folding rise to a map from $M_{[[3]]}$ to this crossed complex, and the only added information in this is the 3-dimensional part $M_{[3]} \to \mathrm{Ker}(\delta)$; this map, we consider as the coboundary of the pair ω, ∇_1. The explicit formula for this coboundary can be read out of the formula (12); on a $P \in M_{[3]}$, say $P = P(x_0; x_1, x_2, x_4)$ with vertices $x_0, \ldots, x_7$, $R(456)$ denotes $R(P(x_4; x_5, x_6))$ etc., and the exponent formation, say the exponent 57 in the second term, denotes the action of $\nabla_1(x_5, x_7) \in G$ on C_2.

7 Integration of differential forms

We present a synthetic theory of integrals which avoids the use of Riemann sums or other approximation techniques and entirely depends on the Fundamental Theorem of Calculus. It also, for simplicity, does not depend on any preorder $\leq$ on the number line R. It depends on one integration axiom, namely: *to any $f : R \to R$, there exists an $F : R \to R$ with $F' = f$; and such F is unique up to a constant.* (The function F is called a *primitive* of f.) With this axiom, one defines one-dimensional integrals: $\int_a^b f := F(b) - F(a)$, for any a, b in R. We can then also define *iterated* integrals: for any $f : R^2 \to R$, one defines $\int_{a_1}^{b_1} \int_{a_2}^{b_2} f$, for any a_1, b_1 and a_2, b_2 in R, and similarly for n-fold iterated integrals.

For simplicity of exposition and geometry, we shall here restrict ourselves to to the case $n = 2$.

Let now M is a manifold and ω a (combinatorial-cubical) 2-form on M.

[2] conventions are a little clumsy here – we might have chosen that $\omega(P(x_0; x_1, x_2) \in C_2(x_0)$, say, but we want to stick to the conventions involved in the construction of the folding operations from [2].

Then we can define a function

$$\int \omega : S_{[2]}(M) \to R$$

as follows. Let $f : R^2 \to M$ be a singular square (= singular 2-cube). By Proposition 13, the 2-form $f^*(\omega)$ on R^2 is of the form

$$f^*(\omega) = \widehat{\theta} \cdot \mathrm{Vol}$$

for a unique function $\widehat{\theta} : R^2 \to R$, and we define

$$\int_f \omega := \int_0^1 \int_0^1 \widehat{\theta}.$$

(We also write the iterated integral in this formula as $\iint_{I \times I} \widehat{\theta}$.) Let us temporarily denote the function $\int \omega : S_{[2]}(M) \to R$ thus defined by the symbol Ω. Then Ω has the properties (i) and (ii) below, which we shall use as definition of the notion of *abstract surface integral*[3]:

(i) $\Omega(f) = \Omega(f') + \Omega(f'')$ whenever f subdivides into f' and f'';
(ii) Ω is alternating in the sense that $\Omega(f \circ \tau) = -\Omega(f)$

where $\tau : R^2 \to R^2$ interchanges the two coordinates. All this is as one would expect; it is standard multivariable calculus. The following, however, has no place in the standard treatment.

Proposition 17 *Let ω be a cubical-combinatorial 2-form on a manifold M, and let $P = P(x; y, z)$ be an infinitesimal parallelogram. Then*

$$\int_{[[x,y,z]]} \omega = \omega(P). \tag{15}$$

Proof. This follows immediately from Proposition 14, since $\iint_{I \times I} c$ for c a constant equals c.

As a Corollary, we have that (cubical-combinatorial) 2-forms ω have the subdivision property: if an infinitesimal parallelogram P subdivides into P' and P'', then $\omega(P) = \omega(P') + \omega(P'')$. For, $\int \omega$ has this property, for all parallelograms, and for infinitesimal parallelograms, $\int \omega$ agrees with ω itself, by (15).

[3]The condition (i) is essentially the same as the defining property of an "observable", as considered in [16], Definition 16.

For an abstract surface integral Ω on M, we may ask whether it extends a given 2-form ω on M, in the sense that

$$\Omega([[x,y,z]]) = \omega(P(x;y,z))$$

for any infinitesimal parallelogram $P(x;y,z)$. The "concrete" surface integral $\int\omega$ does extend ω in this sense, by Proposition 17. We have more completely:

Theorem 18 *For every 2-form ω, $\int\omega$ is the only abstract surface integral extending ω. Thus, there is a bijective correspondence between abstract surface integrals, and 2-forms.*

Proof. By subtraction, this uniqueness assertion is equivalent to the following:

Lemma 19 *If an abstract surface integral Φ on M has the property that it vanishes on all infinitesimal parallelograms, then it vanishes.*

(Recall that an infinitesimal parallelogram $P(x_0;x_1,x_2)$ may be identified with a certain map $[[x_0,x_1,x_2]] : R^2 \to M$.)

Proof. We first note that if $f : N \to M$ is any map between manifolds, an abstract surface integral Φ on M gives rise in an evident way to a an abstract surface integral $f^*(\Phi)$ on N, $f^*(\Phi)(g) := \Phi(f \circ g)$ for any $g : R^2 \to N$. Since f takes infinitesimal parallelograms to infinitesimal parallelograms, we have by the assumption on Φ that $f^*(\Phi)$ vanishes on infinitesimal parallelograms in N.

Now let $f : R^2 \to M$ be an arbitrary element of $S_2(M)$; we have to see that $\Phi(f) = 0$. Since $\Phi(f) = f^*(\Phi)(\mathrm{id})$, and id (the identity map on R^2) is a parallelogram in the affine space R^2 (namely $P(0;e_1,e_2)$, it suffices to see that $f^*(\Phi)$ vanishes on all parallelograms in R^2. In other words, the Lemma follows by taking $\Psi = f^*(\Phi)$ in the following

Lemma 20 *If an abstract surface integral Ψ on R^2 vanishes on all infinitesimal parallelograms, it vanishes on all parallelograms.*

Proof. This will follow if we can prove each of the following three assertions:

1) If Ψ vanishes on all infinitesimal parallelograms, it vanishes on all parallelograms with infinitesimal sides.
(This means parallelograms $P(x;y,z)$ with $x \sim y$ and $x \sim z$, but not necessarily with $x \sim z$.)

2) If Ψ vanishes on all parallelograms with infinitesimal sides, it vanishes on all parallelograms which have (at least) *one* side infinitesimal.

3) If Ψ vanishes on all parallelograms which have one side infinitesimal, it vanishes on all parallelograms.

The proof of 1) is a piece of infinitesimal algebra from [8], (and does not depend on the subdivision property): for fixed x, consider the function $R^2 \times R^2 \to R$ given by $(u, v) \mapsto \Psi(P(x; x+u, x+v))$. It is alternating, because of the alternating property of Ψ, and it vanishes if one of its arguments is 0. Its restriction to $D(2) \times D(2)$ extends therefore to a bilinear alternating map $g : R^2 \times R^2 \to R$. By assumption, g vanishes on $\widetilde{D}(2,2) \subseteq D(2) \times D(2)$. But bilinear alternating maps $R^2 \times R^2 \to R$ are determined by their restriction to $\widetilde{D}(2,2)$, see [8] I. 16.

The proofs of 2) and 3) are quite similar to each other. Let $f(s) := \Psi(P(x; y(s), z)$. Now assume that Ψ satisfies the assumption of 2), and consider a parallelogram $P(x; y, z)$ with the side (x, z) infinitesimal. We use notation $y(s)$, $u(s)$ as in the statement of (2). Then, for $d \in D$,

$$f(s+d) = \Psi(P(x; y(s+d), z)) = \Psi(P(x; y(s), z)) + \Psi(P(y(s); y(s+d), u(s))), \tag{16}$$

by the subdivision property of Ψ . The last term here vanishes because the parallelogram $P(y(s); y(s+d), u(s))$ has infinitesimal sides (note $z - x = u(s) - y(s)$, so $u(s) \sim y(s)$). We conclude $f(s+d) = f(s)$ for all $d \in D$, so $f'(s) = 0$, for all s. Since also $f(0) = \Psi(P(x; x, z)) = 0$ ($P(x; x, z)$ being infinitesimal), we conclude that f is constant 0. Hence $f(1) = 0$, and this assertion is equivalent to $\Psi(P(x; y, z)) = 0$

If we instead had considered a parallelogram with its *first* side (x, y) infinitesimal, we reduce to the case treated just by using the alternating property $\Psi(P(x; y, z)) = -\Psi(P(x; z, y))$.

Finally, assume that Ψ satisfies the assumption in 3). We consider an arbitrary parallelogram $P(x; y, z)$, and define $y(s)$, $u(s)$ as above, and again define $f(s) := \Psi(P(x; y(s), z)$. We then again have the equation (16), and now the last term vanishes by assumption, because $P(y(s); y(s+d), u(s))$ has one side $(y(s), y(s+d))$ infinitesimal. So again we conclude $f'(s) = 0$. Also $f(0) = 0$, since $f(0)$ is Ψ applied to a parallelogram with one of its sides 0 (hence in particular, with an infinitesimal side). So f is constant 0, hence $f(1) = 0$, and this assertion is equivalent to $\Psi(P(x; y, z)) = 0$. This proves 3), and by combining 1), 2) and 3), we get the Lemma.

Proposition 21 *Let Φ be an abstract surface integral on a manifold M. Then there exists a unique 2-form ω on M such that $\Phi = \int \omega$.*

Proof. Uniqueness is clear; for an infinitesimal parallelogram $P = P(x; y, z)$, we must necessarily put $\omega(P) := \Phi([[x, y, z]])$. To see that $\int \omega$ and Φ agree, it suffices by Lemma 19 to see that they agree on infinitesimal parallelograms $[[x, y, z]]$, but ω was defined by that.

Consider the n-cubical groupoid $M_n(R)$. It has transpositions. We have in Section 6 seen that there is a bijective correspondence between alternating connections in it, and n-forms on M (and every connection is automatically alternating). The same recipe which gave this correspondence also gives a correspondence between abstract alternating holonomies in it, and abstract n-dimensional surface integrals.

Proposition 22 *The n-cubical groupoid $M_n(R)$ admits a-integration, i.e. every n-connection in it (automatically alternating) extends uniquely to an alternating holonomy $S_{[n]}(M) \to M_n(R)$. Also, if the n-form ω corresponds to the n-connection ∇,*

$$\int \omega = \mathrm{proj} \circ \int \nabla. \tag{17}$$

Proof. In view of the bijective correspondences which we, for this n-cubical groupoid, have between (alternating) n-connections and n-forms, and between alternating holonomies and abstract n-dimensional surface integrals, this follows, for $n = 2$, immediately from Theorem 18. (The proof in other dimensions is essentially the same.)

Let us finally analyze how a (restricted) version of Stokes' theorem for 1-forms follows from the results and constructions presented. We consider a 1-form ω on M, and the corresponding connection ∇ in $M_1(R)$. We need to be precise about the crossed complex $Cr(M_1(R)')$ correspondingto the 2-cubical groupoid $M_1(R)'$; in dimensions 1 and 2, it is just $M \times R$, and the δ is the identity map. We can safely ignore the M-component, so we shall not consider the folding operations ϕ themselves, but their composite with their projection $\overline{\phi} : M \times R \to R$. In dimension 1, $\overline{\phi}$ itself is just a projection, (recall that $M_1(R)$ in dimension 1 is $M^2 \times R$). In dimension 2, it follows from the Homotopy Addition Lemma [2] Lemma 7.1 (case $n = 1$) that $\overline{\phi}$ to a shell (square of arrows) associates a suitable alternating four-fold sum of the numbers associated to the four arrows of the square. We need one further property of $\overline{\phi}$, namely that it takes composites $+_1$ and $+_2$ in $M_1(R)'$ to sum formation $+$ in R; this follows from [2], (4.9)(i).

So consider a 1-form ω on M, and let ∇ be the connection in $M_1(R)$ corresponding to it by Proposition 15. Then by Proposition 16, $d\omega = \overline{\phi} \circ \widehat{\nabla}$.

Since $\int \widehat{\nabla}$ extends $\widehat{\nabla}$, $\overline{\phi} \circ \int \widehat{\nabla}$ extends $\overline{\phi} \circ \widehat{\nabla}$. Since $\int \widehat{\nabla}$ has the subdivision property, it follows from the quoted equation (4.9)(i) that also $\overline{\phi} \circ \int \widehat{\nabla}$ has the subdivision property. So it has the properties which characterize $\int d\omega$, so that we have the first equality sign in

$$\int d\omega = \overline{\phi} \circ \int \widehat{\nabla} = \overline{\phi} \circ \widehat{\int \nabla},$$

the second equality by the "Formal Stokes' Theorem 10. Let $f \in S_2(M)$ be a singular 2-cube; by the quoted instance of the Homotopy Addition Lemma, applied to the shell $\widehat{\int \nabla}(f)$, $\delta(\overline{\phi}(\widehat{\int \nabla}(f))$ equals $\sum \pm \overline{\phi}(\int_{\partial_i^\alpha(f)} \nabla)$, (standard fourfold sum) which is $\sum \pm \int_{\partial_i^\alpha(f)} \omega = \omega(\partial f)$.

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